# NO 17-PLAYER TRIPLEWHIST TOURNAMENT HAS NONTRIVIAL AUTOMORPHISMS

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Abstract. The existence of triplewhist tournaments for $v$ players has recently been solved for all values of $v$ except $v = 17$. For $v = 12$ and $v = 13$ a complete enumeration has shown the nonexistence of $TWh\,(v)$, while constructions of $TWh\,(v)$ have been presented for $v > 17$. For several values of $v$ existence has been shown by constructing a $TWh\,(v)$ with a prescribed, usually cyclic, automorphism group. In this article it is shown that the strategy of constructing a $TWh\,(v)$ with a prescribed automorphism group cannot succeed with $TWh\,(17)$, since no 17-player triplewhist tournament has nontrivial automorphisms.

## 1. Introduction

A whist tournament, $Wh(v)$, for $v$ players is a schedule of games, each involving two players playing against two others, such that

(1) In each round, the players are partitioned into whist games, with at most one player sitting out (not playing in that round),
(2) each player partners every other player exactly once,
(3) each player opposes every other player exactly twice.

By condition 1, whist tournaments can exist only for $v \equiv 0, 1 \pmod 4$; in fact, whist tournaments exist for all $v \equiv 0, 1 \pmod 4$. A proof can be found in chapter 13 of Anderson's book on combinatorial designs [2].

A whist game is often represented as an ordered 4-tuple $(n, e, s, w)$, where $n$, $e$, $s$, and $w$ represent the players sitting on the north, east, south, and west side of the playing table. Partners sit opposite to each other; thus, the partner pairs are $\{n, s\}$ and $\{e, w\}$. For the purposes of this article, we ignore the actual seating of the players and consider two whist games identical if they have the same pairs of partners.

In a triplewhist game $(n, e, s, w)$ the pairs $\{n, e\}$ and $\{s, w\}$ are opponents of the first kind, and $\{n, w\}$ and $\{s, e\}$ are opponents of the second kind. As for whist games, we consider two triplewhist games identical if they have the same pairs of partners, opponents of the first kind, and opponents of the second kind, respectively.

A triplewhist tournament, $TWh(v)$, for $v$ players is a whist tournament with the additional condition

(4) each player has every other player once as an opponent of the first kind, and once as an opponent of the second kind.

The existence of $TWh\,(v)$ is only open for $v = 17$. Lu and Zhu [8] show that a $TWh\,(v)$ exists for all sufficiently large $v \equiv 0, 1 \pmod 4$, leaving only 15 cases in the range $12 \leq v \leq 133$ open. Haanpää and Östergård [7], as well as Ge and Lam [4], find that no $TWh\,(12)$ exists; Haanpää and Kaski [6] find that no $TWh\,(13)$ exists; and the remaining open cases with the exception of $v = 17$ have been settled by Ge and Zhu [5], Ge and Lam [4], and most recently by Abel and Ge [1].

## 2. Automorphisms of $TWh\,(v)$

We consider a $TWh(v)$ to be a set of rounds, each of which is a set of triplewhist games. We denote the set of players in a $TWh\,(v)$ by $V$ and let the symmetric group $S_{|V|}$ act on $V$ in the natural way. This induces an action on the triplewhist games, rounds, and triplewhist tournaments. A permutation $g$ is an automorphism of the triplewhist tournament if it maps the triplewhist tournament onto itself. For $g \in S_{|V|}$ we define $V_g = \{v \mid v^g = v, v \in V\}$ as the set of those players in $V$ fixed by $g$.

**Lemma 1.** *If an automorphism $g$ of a $TWh\,(17)$ fixes the pair of players $\{v_1, v_2\}$, then $g$ also fixes the player sitting out in the rounds where $v_1$ and $v_2$ play as partners, opponents of the first kind, or opponents of the second kind.*

*Proof.* Consider the round $R$ where $v_1$ and $v_2$ play as partners (or as opponents of the first or second kind, respectively). Then $v_1$ and $v_2$ also play as partners in round $R^g$. Since $v_1$ and $v_2$ only play as partners in one round, we have $R = R^g$. It follows that $g$ must fix the player sitting out. □

**Lemma 2.** *For all automorphisms $g$ of a $TWh\,(4k + 1)$, if $|V_g| \geq 2$ then $|V_g| \geq 5$.*

*Proof.* Let $g$ fix the players $v_1$ and $v_2$. By Lemma 1, since $g$ fixes $\{v_1, v_2\}$, $g$ must also fix the player sitting out in the rounds where $v_1$ and $v_2$ play as partners, opponents of the first kind, and opponents of the second kind. These rounds are distinct. □

**Lemma 3.** $|V_g| \leq v/4$ *for all nontrivial automorphisms $g$ of a $TWh\,(4k + 1)$.*

*Proof.* Suppose $|V_g| > v/4$. On one hand, consider the rounds in which some two players fixed by $g$ play at the same triplewhist game. By Lemma 1, the player sitting out in those rounds must be a fixed point under $g$. On the other hand, consider the rounds in which no two players fixed by $g$ play at the same triplewhist game. There are only $\lfloor v/4 \rfloor$ triplewhist games in each round, so by the pigeon hole principle the player sitting out must be a fixed point. Since in every round the player sitting out is a fixed point, all players are fixed $g$; thus, $g$ is trivial. □

**Lemma 4.** *No nontrivial automorphism $g$ of a $TWh\,(17)$ fixes more than one point.*

*Proof.* This follows immediately from Lemma 2 and Lemma 3. □

**Lemma 5.** *A $TWh\,(17)$ cannot have an automorphism $g$ that would consist of eight 2-cycles and a fixed point.*

*Proof.* Suppose that such a $g$ would exist and choose some $v_1$ and $v_2$ such that $v_1^g = v_2$ and $v_2^g = v_1$. Since $g$ fixes $\{v_1, v_2\}$, by Lemma 1 it fixes the player sitting out in the rounds where $v_1$ and $v_2$ play as partners, opponents of the first kind, or opponents of the second kind. These three players are distinct fixed points of $g$, a contradiction. □

**Theorem 1.** *No $TWh\,(17)$ has nontrivial automorphisms.*

*Proof.* Let $g$ be a nontrivial automorphism of a $TWh\,(17)$, let $n$ be the order of $g$, and let $p$ be a prime that divides $n$. Now $g' = g^{n/p}$ is a permutation of prime order $p$. Such a permutation must consist of one or more $p$-cycles and possibly a number of fixed points; clearly $2 \leq p \leq 17$. By Lemma 4 we find that $p \notin \{3, 5, 7, 11, 13\}$, since no permutation that would consist of one or more $p$-cycles and at most one fixed point can exist when $17 \not\equiv 1 \pmod{p}$. The case $p = 17$ can be eliminated by noting that Finizio [3] reports that there is no $\mathbb{Z}$-cyclic $TWh\,(17)$. Thus, $g'$ must be of order 2; to satisfy Lemma 4 it must consist of eight 2-cycles and a fixed point. However, Lemma 5 shows that this is impossible. □

## 3. Conclusions

The question of the existence of a $TWh\,(17)$ remains open. The task of carrying out an exhaustive search for a $TWh\,(17)$ appears to be well beyond the reach of current computers and techniques. It is a most natural idea to limit the search space by limiting the search to structures with a prescribed automorphism group, and this technique has been used with good effect in proving the existence of $TWh\,(v)$ for several values of $v$; indeed, for many values of $v$ the existence of a $TWh\,(v)$ has been proven by constructing a $TWh\,(v)$ with cyclic symmetry. The result in this article shows, however, that this technique cannot work in the case of $TWh\,(17)$, as no $TWh\,(17)$ has nontrivial automorphisms.

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